AP Calculus Summer Research

AP Calculus Course at a Glance

Greetings to those who have signed up for AP Calculus AB course offering at White Station H.S. This is the very first math course that will be challenging to most of you because your successful study habits in prior courses will not work for this course. For example, Algebra II based on my experience, appears to be easy for those who can watch the teacher give two examples of a type of problem, look up three more examples in the book and on-line, then be a good parrot and repeat the same steps. Boring! In contrast, calculus requires that you understand a concept before you apply mathematical strategies. A famous quote, often associated with Albert Einstein, "If you cannot explain it, then you do not understand it." You must learn to explain verbally before you attack a problem analytically. You must be familiar with each problem set in a four-dimensional space- *(i) verbally, (ii) graphically, (iii) numerically, and (iv) functionally*.

Step 1: Locate and download the AP Calculus course at a glance that has been posted by the College Board at https://apcentral.collegeboard.org/media/pdf/ap-calculus-ab-bc-course-at-a-glance_0.pdf . Note that the parts shaded in violet including units 9 and 10 are for BC Calculus only.

Step 2: Read the description of parts of Unit 1. The two big ideas in Unit 1 are the application of *limits and continuity/discontinuity*. According to Ms. Wong, both ideas were introduced in the second half of 4^{th} quarter of your Pre-Calculus course. Summer work will focus on keeping those two ideas fresh in your mind as we begin school. Expect a great deal of AP Classroom practice.

Concepts for Limits (part I)- Remember that finding a limit is about how a function approaches a value and not necessarily making it to that value. We like to say, **"*When it comes to limits, it is about approaching there rather than getting there.*"** Analytically, limits address the idea approaching an f(x) value at a given number for x rather than getting to equal an f(x) value.

Be familiar with the following areas of limits when we begin the course-
(i) writing limit notation,
(ii) writing one-sided limit notation,
(iii) using limits with x approaching $\pm\infty$ to locate horizontal asymptotes,
(iv) using limits to get vertical asymptotes.

Concepts for Continuity (Part II)-

Be able to verbally explain how we use limits to determine continuity at a point and provide one example that is continuous and three different examples of discontinuity. Identify the four types of discontinuities verbally, graphically, and analytically. Try a search engine for images. I found this link as an example- https://calcworkshop.com/limits/limits-and-continuity/

Calculator recommendation for this course: We will have TI Inspires available in our classroom. I think that the TI 84 + is faster because there are fewer submenus with less places to hit the wrong button. If you are shopping, try to get a TI 84+ or TI 84 +CE if possible. You can stay with TI Inspire if you already prefer TI Inspire.

Part I (Limits)

Examples of Limits

Instructions: Go online and use DESMOS graphics to consider the following functions. Be sure that DESMOS is in the "RADIAN" mode and not degrees.

1. Consider $y = (x-1)^2 + 2$
   (i) Find $\lim_{x \to 1} (x-1)^2 + 2$
   From the DESMOS graph, you can see that the limit is y = 2.

   If you do not have graphing technology available, the analytical approach has several strategies. The fastest, riskiest strategy is to merely substitute the value of x into the equation to see if you get something qualitatively acceptable. Sometimes, this fails, and other ideas must be implemented. See the follow-up examples. **Strategy #1- Direct substitution**

2. Find $\lim_{x \to 1^+} (x-1)^2 + 2$ This is a one-sided limit where you consider approaching from the right-hand side of the function. Let's look at a numerical approach to this one-sided limit. What value of y do you get for the x values of x = 1.1, 1.01, 1.001, 1.0001?

When x =	1.1	1.01	1.001	1.0001	1
Then y =	2.01	2.0001	2.000001	2.00000001	2

A similar limit exists from the left. You can see from the table that the value of y is approaching 2 as the value of x gets closer to 1.

Find $\lim_{x \to 1^-} (x-1)^2 + 2$ This is a one-sided limit where you consider approaching from the left-hand side of the function. Let's look at a numerical approach to this one-sided limit. What value of y do you get for the x values of x = 0.9, 0.99, 0.999?

When x =	0.9	0.99	0.999	1
Then y =	2.01	2.0001	2.000001	2

Strategy #2- Numerical approach using x values increasingly closer to the specific x

Approaching the same limit from both left and right is a necessary condition for a function to be continuous. However, there are other requirements before continuity is confirmed. A hole for example would satisfy the limit condition yet is not continuous. See the next example.

3. Consider $\lim_{x \to 2} \frac{x^2+x-6}{(x-2)}$. If you attempt the simple strategy of substitution of x = 2 into the function, you get a strange result of $\frac{0}{0}$. This is known as "*an indeterminate form*".

If you encounter (any number except 0)/0, the function is undefined or does not exist (DNE).
If you get 0/(any number except 0), the value is zero.
The indeterminate ratio of 0/0 has possibilities of (DNE), a finite value, or 0.
You must try something else. There are several options for limits approaching $\frac{0}{0}$.

Strategy # 3 for limits applies when you see a rational function with an indeterminate limit. Try dividing out the denominator.

$$\lim_{x \to 2} \frac{x^2+x-6}{(x-2)} = \lim_{x \to 2} \frac{(x-2)(x+3)}{(x-2)} = \lim_{x \to 2} x+3 = 5$$

A numerical approach reveals the same limit from both sides. See below

x =	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x) =	4.9	4.99	4.999	5	5.001	5.01	5.1

Graphing in DESMOS or your calculator will reveal a straight line with a hole at x = 5. Although limits from both sides exist and are equal, this function is not continuous.

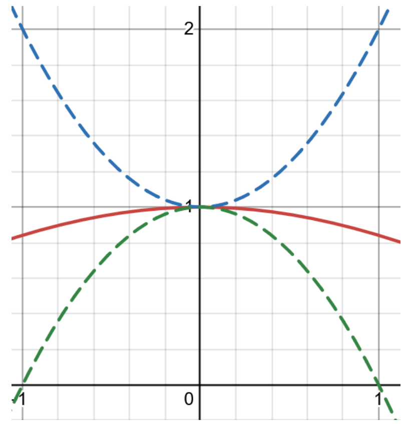
4. **Strategy #4, *Rationalizing Technique***, can be employed for some functions. This technique may be employed when multiplication by 1 = (conjugate)/(conjugate) cleans up the function. Consider $\lim_{x \to 0} \frac{\sqrt{(x+1)}-1}{x}$
First, try strategy 1, direct substitution. You get the indeterminate form of 0/0. Strategy #2, tabular construction takes time. Strategy 3 does not work out well either. You can multiple the numerator and denominator by the conjugate of $\sqrt{(x+1)}-1$ which is $\sqrt{(x+1)}+1$.

$$\lim_{x \to 0} \frac{\sqrt{(x+1)}-1}{x} = \lim_{x \to 0} \frac{\left[\sqrt{(x+1)}-1\right](\sqrt{(x+1)}+1)}{x(\sqrt{(x+1)}+1)} = \lim_{x \to 0} \frac{(x+1)-1}{x(\sqrt{(x+1)}+1)} = \frac{1}{2}$$

5. **Strategy #5** is ***The Squeeze Theorem***. Consider $\lim_{x \to 0} \frac{\sin(x)}{x}$. Once again, the indeterminate form appears. Before showcasing the Squeeze Theorem, let's look at a numerical approach. The table below shows the limit to be 1 from both sides.

f(x)	0.84147	0.99833	0.9999998		0.9999998	0.99833	0.84147
x	-1	-0.1	-0.01	0	0.01	0.1	1

The Squeeze Theorem (also known as the Sandwich Theorem) is a neat concept. Consider three functions. The function in question from the limit, $f_L(x)$ at a point where x = some specific value, say x = c and you want the limit as x → c. For the current example x → 0.



Suppose that we have a continuous function f_{high} that is *above* f_L *everywhere except at c.* Example, $f_{high} = 1 + x^2$ will do. Also suppose that we have a continuous function f_{low} that is below f_L *everywhere except at x = c. For example,* consider $f_{low} = 1 - x^2$. See the image of all three functions in the image to the right.

The Squeeze Theorem concept is that if the upper and lower functions converge to the same limit and if the function in question is always in between the upper and lower functions, then the function in question must also approach the same limit.

Since $\lim_{x \to 0} f_{high} = \lim_{x \to 0} f_{low} = 1$, the limit for the function in between must also be x = 1.

We will learn another way to find limits of indeterminate forms after chapter 2. We will use something known as L'Hôpital's Rule. This is later in the course.

<u>Special Case Limits to Know</u>

The previous example brings to this course one of three special limits that you need to know.

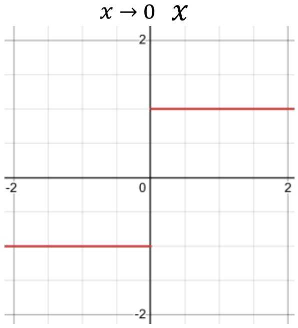
$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$	$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$	$\lim_{x \to 0} (1 + x)^{1/x} = e$

Use Desmos to graph each of these functions to check the appearance of the limit in each case.

<u>Three Common Limits That Fail to Exist</u>

Jump Discontinuity

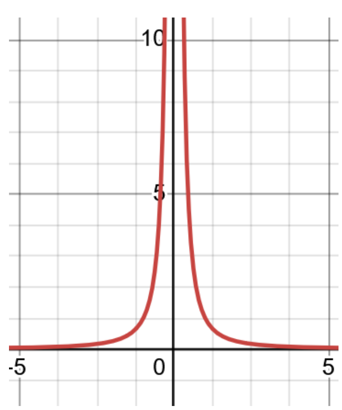
$$\lim_{x \to 0} \frac{|x|}{x}$$



The limit from the left is not the same as the limit from the right.

Unbounded Behavior

$$\lim_{x \to 0} \frac{1}{x^2}$$



Oscillating Behavior

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

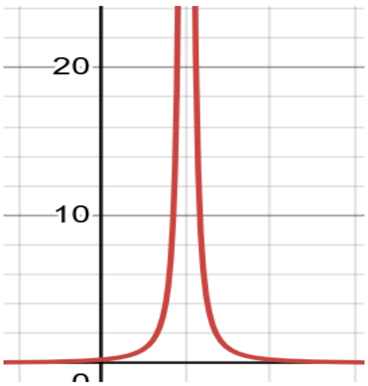
Graph this with DESMOS to confirm that it never settles down at the origin.

Try a tabular approach using value of $x = \frac{2}{(n\pi)}$ where n is the list of odd numbers, such as n = 1, 3, 5, 7, 9, …

Vertical Asymptotes

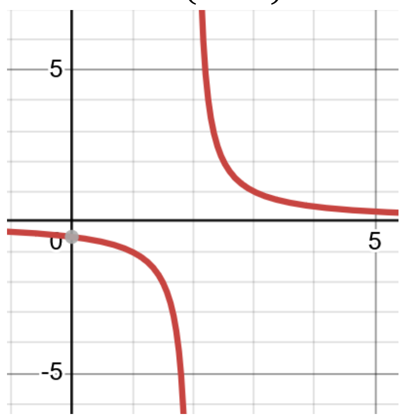
The middle example on the previous page is an example of a vertical asymptote. There is an interesting conflict in classification of limits when it comes to vertical asymptotes. In some places the limit is classified as "Does Not Exist" or DNE. In other places, the limit can be also classified as ± ∞. What gives? Most importantly, what does The College Board think? Consider the following three examples with graphs.

$$\lim_{x \to 2} \frac{1}{(x-2)^2}$$



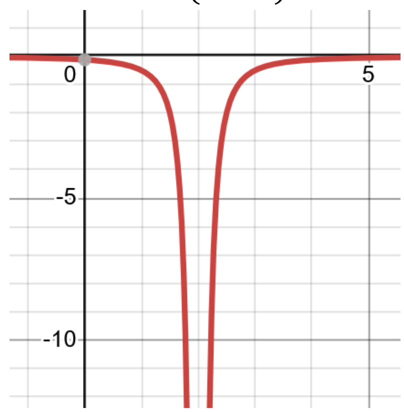
Since both sides approach +∞ the limit is +∞. Choose this over DNE on AP exam.

$$\lim_{x \to 2} \frac{1}{(x-2)}$$



The limit from the left approaches -∞. The limit from the right approaches +∞. The limit does not exist.

$$\lim_{x \to 2} \frac{1}{2(x-2)^2}$$



Both sides approach -∞ as x approaches a value of 2. Choose the limit is -∞ instead of DNE if given the choice.

Always look for vertical asymptotes by finding the limit when the denominator approaches zero while the numerator approaches a non-zero value.

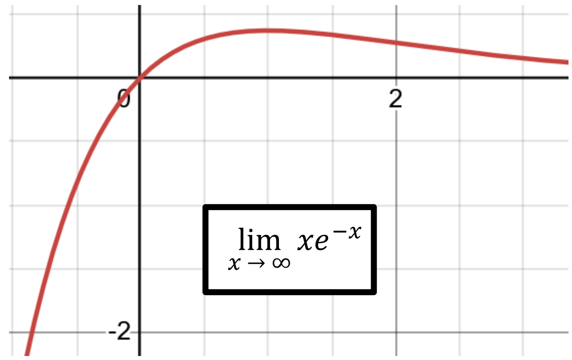
Limits at x → ± ∞

There are several obvious limits for transcendental functions. Be aware of the graphs for sin(x), cos(x), tan(x), sec(x), csc(x), cot(x), ln(x), and $e^{\pm x}$. I have found that some students need a review, in particular of the last two functions. I am personally rusty about sketching the graphs of the co-functions of trigonometry. Take note of the limits of each graph by graphing these in DESMOS. Consider some of the following limits.

$$\lim_{x \to 0} \ln(x) \qquad \lim_{x \to \infty} \ln(x) \qquad \lim_{x \to 0} e^x \qquad \lim_{x \to \infty} e^x \qquad \lim_{x \to -\infty} e^x \qquad \lim_{x \to \infty} xe^{-x}$$

The last example can be rewritten as a fraction $\frac{x}{e^x}$. The numerator, (x), is approaching infinity while the denominator, e^x, is also approaching infinity. You are looking at our first encounter of the second type of *indeterminate form*, $\frac{\infty}{\infty}$. We can apply L'Hôpital's Rule to this form as well but only after we have mastered some skills in units 2 and 3. Let's explore this limit on the next page.

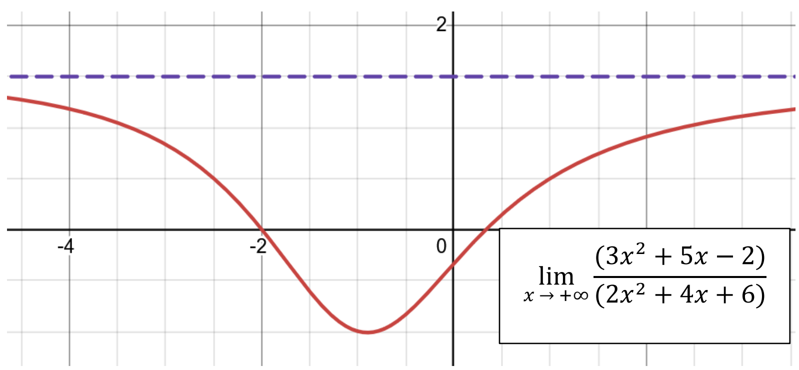
This course expects each student to be able to explain and solve problems verbally, analytically, graphically, and numerically. We can graph the function if this is a calculator friendly problem, but two-thirds of your AP exam is not calculator friendly. We could make a table which would once again require a calculator. We could analytically solve this problem using a technique that we will see later in the course. However, we can already recognize the solution verbally. The above graph is for reference.



Verbal Explanation and justification uses (i) Claim, (ii) Evidence, and (iii) Reasoning-
Claim- The limit is zero.
Evidence- Exponential functions approach infinity faster than a linear function.
Reasoning- Rewriting as a fraction, the denominator approaches large values faster than the numerator making the ratio increasingly less than one as x increases from zero.

Practice Problem- Can you verbally explain the following limit? $\lim_{x \to -\infty} xe^{-x}$

Rational Functions are a prime target for testing limits at plus or minus infinity. Consider the limit for the graph shown in the image to the right. The dashed line shows a horizontal asymptote as x approaches $\pm\infty$. How would you address either of these limits *analytically*? *verbally*?



Considering the direct substitution method leads to the indeterminate form of $\frac{\infty}{\infty}$. The analytical approach with indeterminate forms is to use L'Hôpital's Rule. Another analytical technique is to try long division which leads to 3/2 plus a remainder. Long division shows the rational function reducing to $\frac{3}{2} + \frac{-(x+11)}{2x^2+4x+6}$. The limit as x approaches infinity for this fraction approaches 1.5 + 0.

The verbal explanation is shown below.
Claim: The limit as x approaches positive infinity is 3/2.
Evidence: The highest degree term for numerator and denominator are to the same power.
Reasoning: Since the highest degree terms are the same power then the limit will be the ratio of the coefficients of the highest degree terms. This holds for the limit towards negative infinity.

The general rule for limits approaching infinity for rational functions can be stated without a need of L'Hôpital's Rule through justification of long-division.

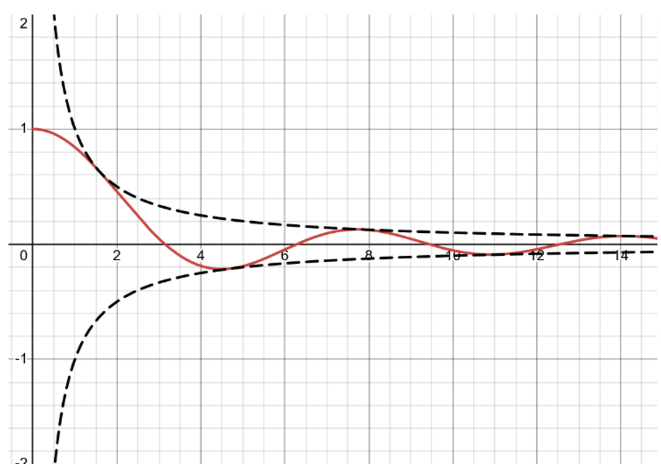
Limits approaching infinity of rational functions can be stated based on consideration of the highest degree terms in numerator and denominator.

- If the highest degrees are equal, then the limit will be the ratio of the leading coefficients.
- If the higher degree is in the denominator, then the limit will be zero.
- If the higher degree is in the numerator, then the limit will be ∞ or DNE.

Horizontal asymptotes are determined using limits approaching $\pm\infty$.

Limits at infinity for trigonometric functions. Sine and cosine oscillate between +1 and -1 and never reach a specific value. What happens to limits when these are used in a ratio? Consider for example the following limit. $\lim\limits_{x \to \infty} \frac{\sin(x)}{x} = ?$.
The verbal explanation is shown below.
Claim: The limit as x approaches positive infinity is 0.
Evidence: The numerator remains as a value somewhere between -1 and 1 while the denominator approaches an infinite value
Reasoning: The division of a finite value by a large number produces a small number. As the larger value in the denominator increases continually while the numerator remains small, the ratio continues to shrink in value, approaching zero.

Your textbook uses the Squeeze Theorem to demonstrate this graphically and analytically. Consider the following high/low functions to sandwich our ratio. $\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$. See figure to the right.



Both high and low functions approach zero as x approaches infinity. According to the Squeeze Theorem, the limit of our $\frac{\sin(x)}{x}$ must also approach zero.

Continued on Next Page

Part II: Continuity

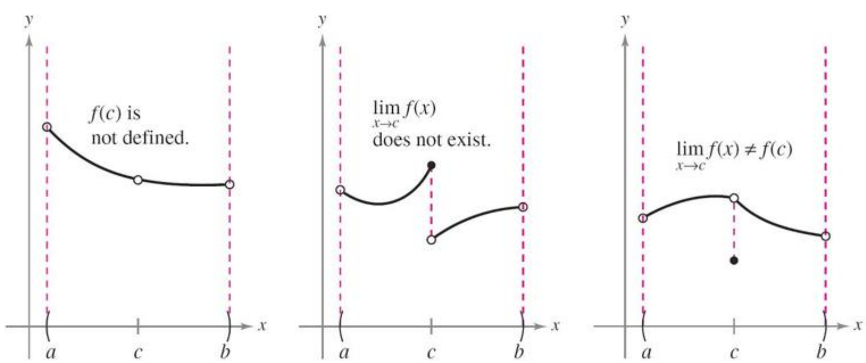
Continuity at a point requires three conditions. Failure of any of the three conditions implies failure of continuity at a point. Your textbook prefers to identify a specific point with the letter "c", as in a constant value. When the variable x is designated to be a specific value of c, then *f(x)* is written *f(c)*.

1. The function is defined at x = c. In other words, f(c) exists. No hole at f(c).
2. The limit exists at x = c. In other words, the function approaches the same value from both sides. This is assuming that the point in question is not at an endpoint of an interval. More on endpoints when class begins.
3. The limit as x approaches c is the same as the value of *f(c)*.

Three Common Examples of Discontinuity

(i) The function is not defined at x = c such as a hole.
(ii) The limit approaches different values from left and from right such as a jump discontinuity.
(iii) The limit exists as the same value from both sides and the function exists; however, the value of the limit is not the same as the value of the function. See figures below.

The image below is from your textbook.



The Intermediate Value Theorem

Think about the label. What does intermediate value mean? It is a number between the endpoints. **If** a function is continuous on the interval between x = a and x = b, and **if** the height of the end points *f*(a) and *f*(b), then there will be at least one x that creates every y value in between the high and low values of *f*(a) and *f*(b). This assumes the *f*(a) ≠ *f*(b). Image below is from textbook.

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

You can abbreviate IVT for a justification on the AP exam. Be sure that you have continuity before you apply IVT to a justification.

Summer Assignment

Bring to class on the first day of school the following:

1. Your fully charged calculator, if you have one.

2. A sheet of paper or two with hand-written notes of what you believe to be the twelve most important ideas listed on the previous 8 pages. More than 12 will be acceptable.

   Students who write by hand retain more than students who type or look-over to take mental notes. Mental notes don't work anyway because I need you to submit your work for a grade.

I plan to have each student share an idea with our class until we exhaust all ideas. Bring something to write with as you may want to use somebody else's ideas to enhance your own list.

Conclusion

You have been told throughout your entire mathematical growth and development that you cannot divide by zero. Yet, here, we are confronted with the notion of $\frac{0}{0}$ as an interesting limit. We are still not dividing by zero but rather *looking at math as a function is moving* in the direction towards this limit.

The most fascinating thing about calculus as compared to all other math courses that you have taken thus far is that *calculus is kinetic math*. We consider what happens as x moves along an axis in a microscopic sense. This is the first time that I truly began to appreciate the power of math. No longer boring and static!

There are two amazing ideas concerning the application of the limit. The first big idea is the *derivative* as a limit. Unit 2 begins our journey with the introduction of the derivative to applications in unit 5.

The second big idea involves *antiderivatives and integration*. Again, the limiting process is the key. Again, we deal with some kinetic math. Units 6 – 8 take a look at something that is a most powerful process for all of science, medicine, and engineering.